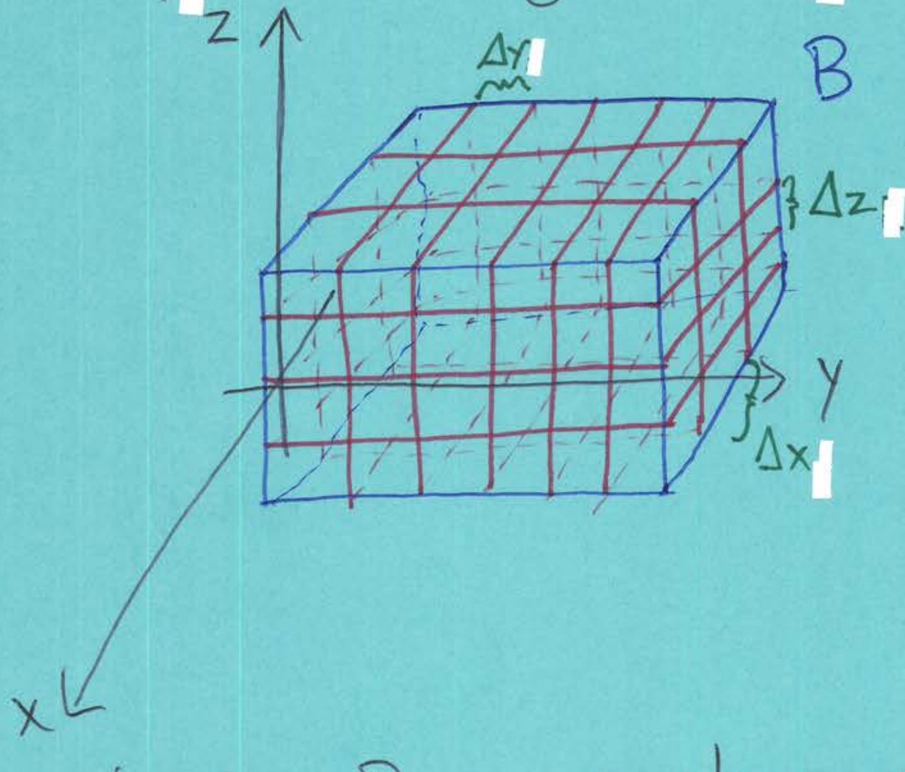
Lecture 27

15.7 - Triple Integrals

We do triple integrals over solid regions in $\mathbb{R}^3$. Let's start with the simple case of a box $B = [\alpha, \beta] \times [\gamma, \delta] \times [\varepsilon, \zeta]$. We cut this box into tiny boxes B_{ijk} of length Δx, width Δy, and height Δz.



In each box B_{ijk}, we choose a sample point (x_i^*, y_j^*, z_k^*).

Then the triple integral of $f(x,y,z)$ over B is given by the Riemann sum

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\Delta V$$

where $\Delta V = \Delta x \Delta y \Delta z$.

Thus we have:

$$\iiint_B f(x,y,z)\,dV = \int_{\varepsilon}^{\zeta}\int_{\gamma}^{\delta}\int_{\alpha}^{\beta} f(x,y,z)\,dx\,dy\,dz$$

-or- any of the other 5 permutations:
$dxdzdy$, $dydxdz$, $dydzdx$
$dzdxdy$, or $dzdydx$
with the appropriate bounds of integration.

Let's do a basic example:

Ex: Compute the integral of $f(x,y,z) = xy\cos z$ over the box $B = \{(x,y,z) \mid 0 \le x \le 4, 0 \le y \le 2, -\frac{\pi}{2} \le z \le \frac{\pi}{2}\}$

Sol:

$$\iiint_B xy\cos z \, dV = \int_0^4 \int_0^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xy\cos z \, dz\, dy\, dx$$

$$= \int_0^4 \int_0^2 xy\sin z \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy\, dx$$

$$= \int_0^4 \int_0^2 \left(xy - xy(-1)\right) dy\, dx$$

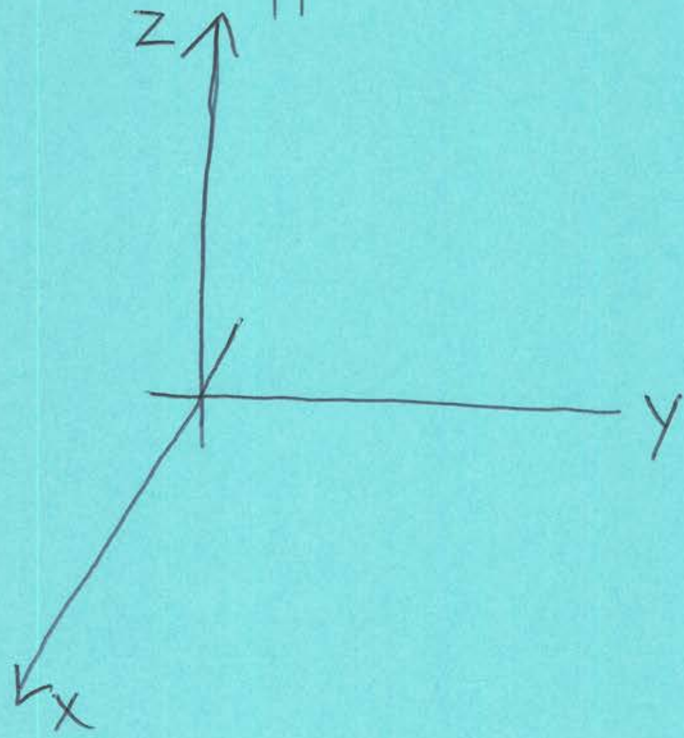
$$= \int_0^4 \int_0^2 2xy\, dy\, dx = \int_0^4 xy^2 \Big|_0^2 dx$$

$$= \int_0^4 (4x - 0)\, dx = \int_0^4 4x\, dx = 2x^2 \Big|_0^4 = \boxed{32}$$

$\Diamond$

As with double integrals, we don't need to stick to the case of a box, we can integrate over more general "elementary regions".

When setting up bounds for general regions, E the process will be similar to the double integral case. Suppose we orient our axes like so:



Then, if we start by integrating:

- x: we go from the "back function" to the "front function"
- y: we go from the "left function" to the "right function"
- z: we go from the "bottom function" to the "top function".

Once again, it is important to **sketch the region of integration**!

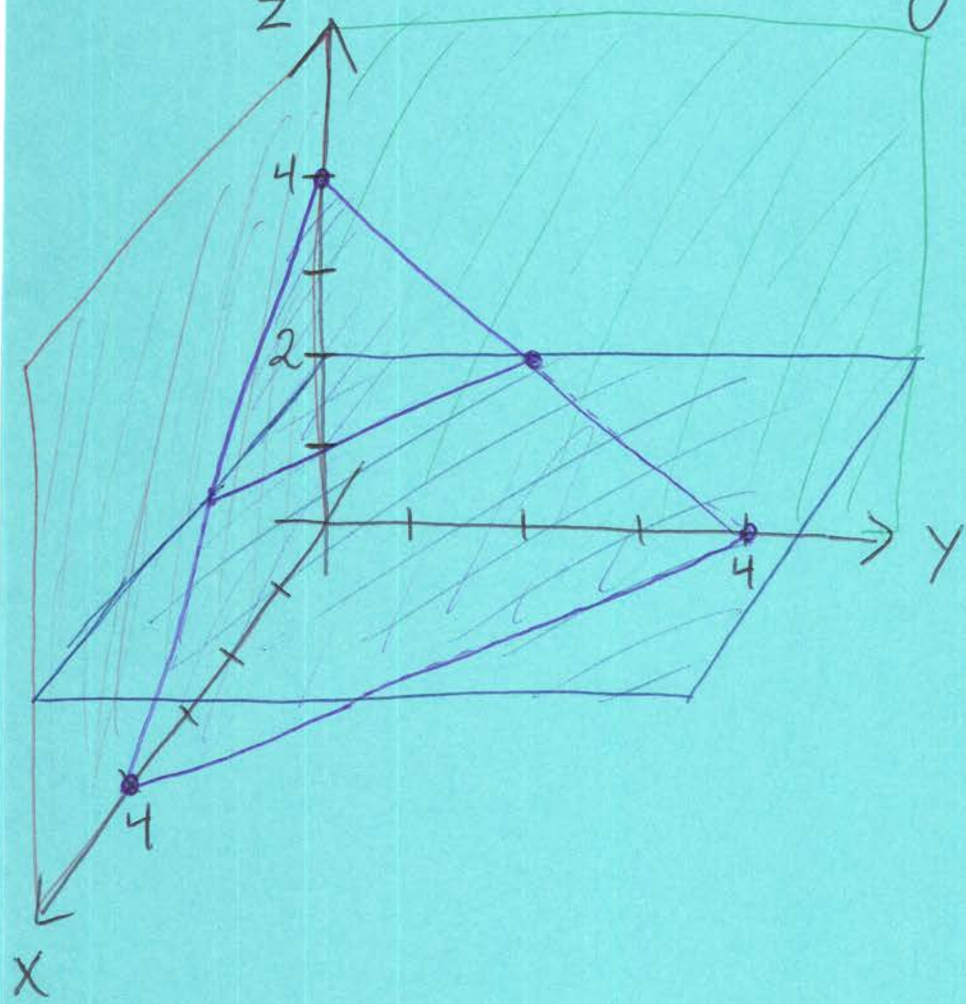
Once we have the bounds on the inside integral, we get the bounds on the outer two as follows if you started by integrating:

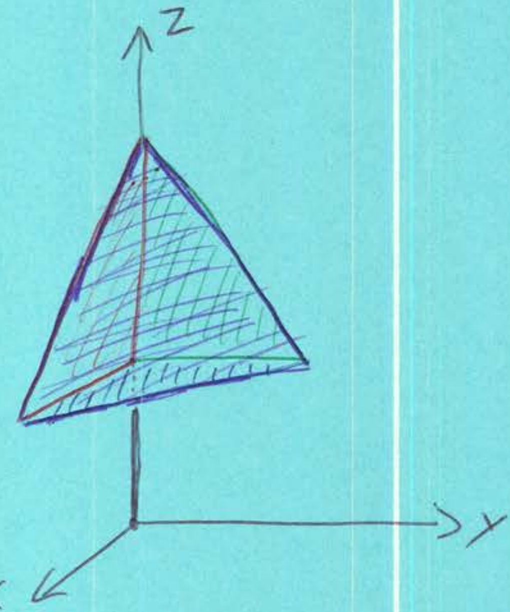
- x: look at the "shadow" of E in the yz-plane, and set up a double integral over that region
- y: look at the "shadow" of E in the xz-plane, and set up a double integral over that region
- z: look at the "shadow" of E in the xy-plane, and set up a double integral over that region.

Let's see an example to make this concrete.

Ex: Set up the interated integral to compute $\iiint_E z\,dV$ where E is the tetrahedron bounded by the planes $x=0, y=0, z=2$, and $x+y+z=4$.

Sol: First, sketch the region:





$x=0$ is the yz-plane & $y=0$ is the xz-plane

$z=2$ is parallel to the xy-plane, moved up 2.

$x+y+z=4$ passes through the points $(4,0,0)$, $(0,4,0)$, and $(0,0,4)$, the three of which determine the plane.

Let's integrate y first, because "why not?"

If we integrate y first, we start at the left function and go to the right function.

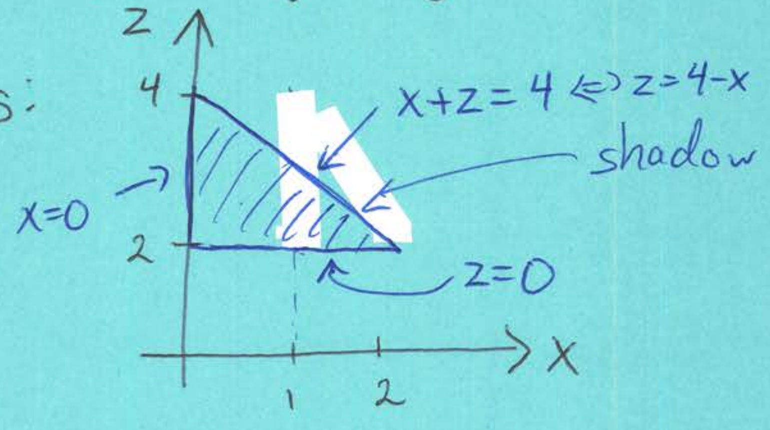
The left function is the red plane: $y=0$

The right function is the purple plane: $x+y+z=4$
-or-: $y=4-x-z$.

So, the inside integral is: $\int_0^{4-x-z} z\,dy$

To get the outside 2, we must integrate over the shadow in the xz-plane. You can think of this shadow as just squishing E into the xz-plane.

This gives us:



If we integrate z next, then x, we have:

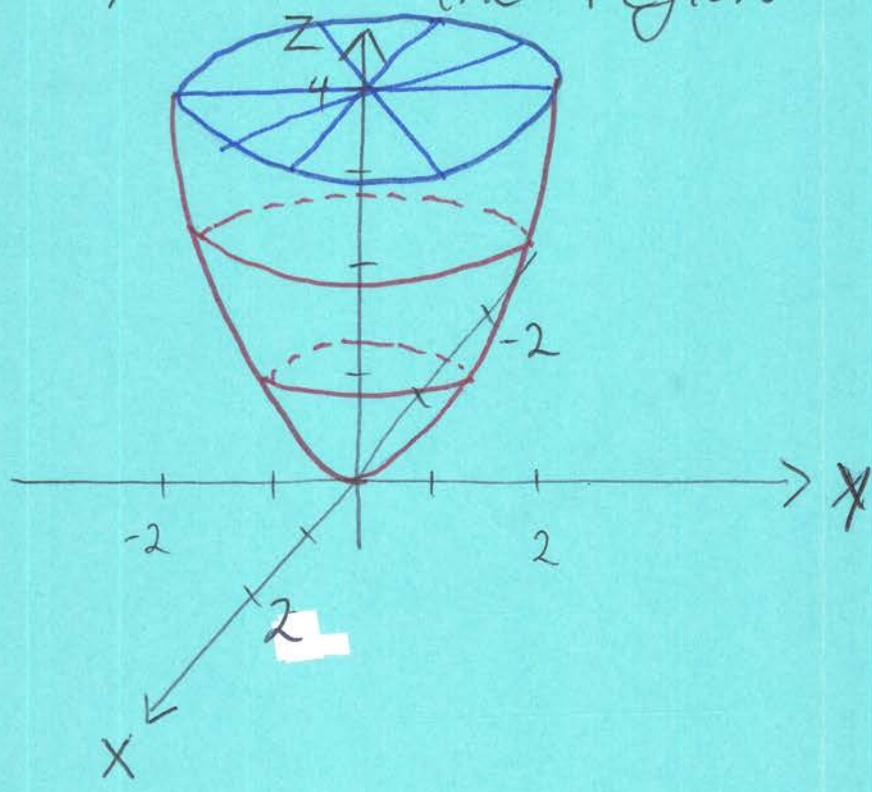
$$\iiint_E z\,dV = \int_0^2 \int_2^{4-x} \int_0^{4-x-z} z\,dy\,dz\,dx.$$

◇

Just as $\iint_D 1\,dA$ gave the area of a region D, $\iiint_E 1\,dV$ gives the volume of the solid E.

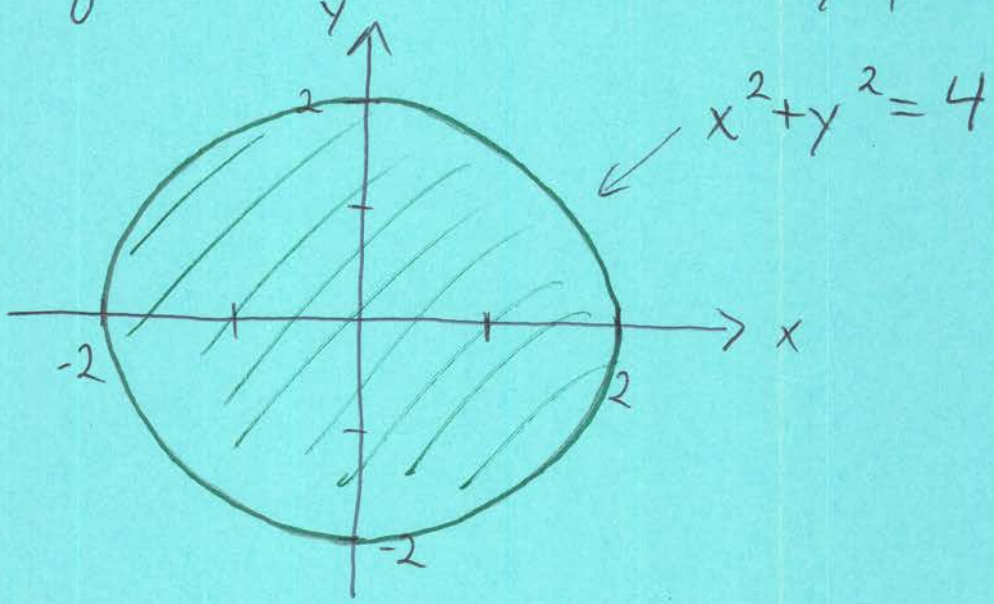
Ex: Find the volume of the solid bounded by $z=x^2+y^2$ and $z=4$.

Sol: First, sketch the region:



Let's integrate z first: the bottom function is $z=x^2+y^2$ and the top is $z=4$.

If we squish this onto the xy-plane, we have:



the disk of radius 2. If we integrate with respect to x next, we have that the lower bound is $x=-\sqrt{4-y^2}$ and the upper is $x=\sqrt{4-y^2}$ so the bounds on y are $-2 \leq y \leq 2$. Thus

$$\text{Vol}(E) = \iiint_E 1\,dV = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{4} dz\,dx\,dy.$$

◇

This integral can be done in a much easier way. Think about it. <u>Hint</u>: it involves polar coordinates...

Sometimes, we even need to switch the order of integration for triple integrals. Again, you'll need to be able to read the region of integration from the bounds of the integral.

Ex: Change the order of integration in

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

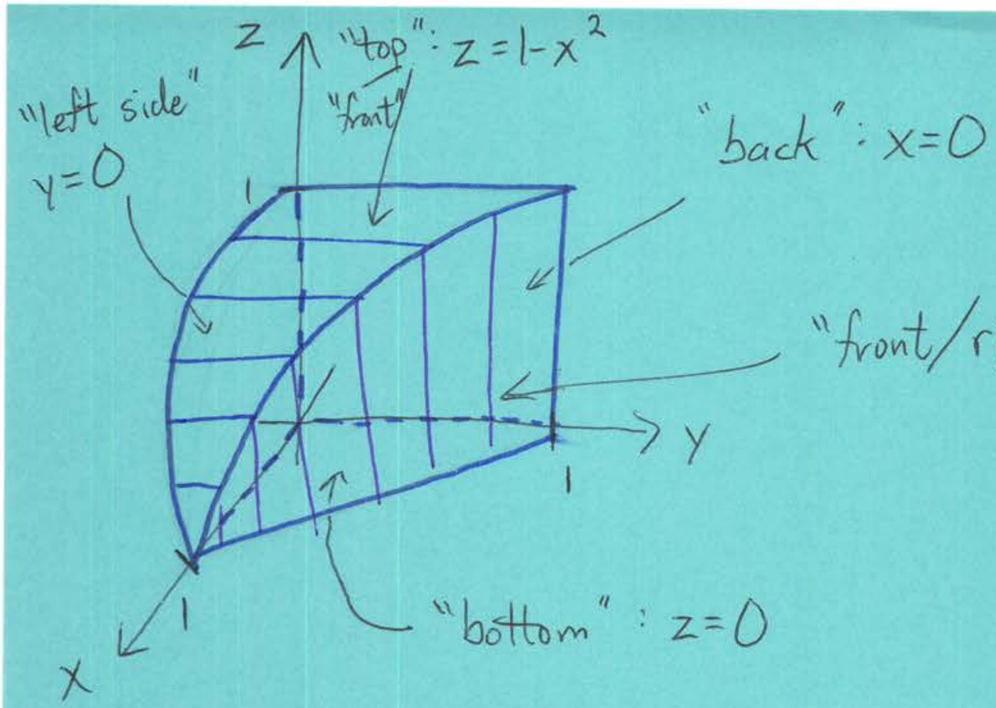
to something else.

Sol: Sketch the region:

The bounds give:
- $0 \le y \le 1-x$
- $0 \le z \le 1-x^2$
- $0 \le x \le 1$

The region is bounded by:

$y=0,\ y=1-x,\ z=0,\ z=1-x^2,\ x=0,\ x=1$

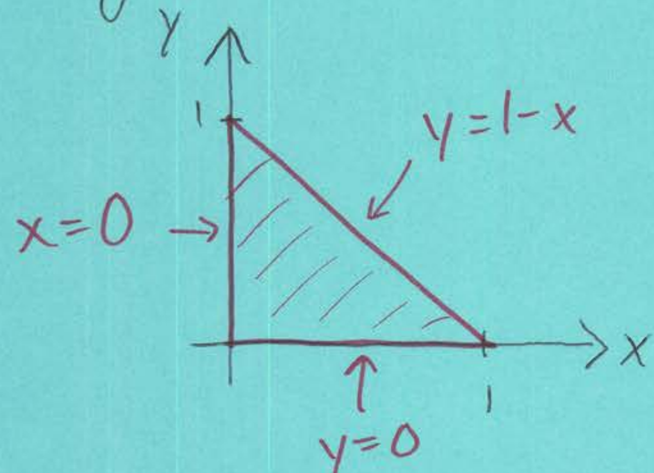


Let's do an integral $dz\,dx\,dy$:

Integrating z first means "bottom" to "top":

$$0 \le z \le 1-x^2$$

Squishing this to the xy-plane gives a triangle:



So, the integral is

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx = \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy$$

$\diamond$